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A direct proof of the CSW rules

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Abstract: Using recursion methods similar to those of Britto, Cachazo, Feng and Witten (BCFW) a direct proof of the CSW rules for computing tree-level gluon amplitudes is given.

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Contents

1. Introduction

Over the last 18 months there has been significant improvement in the methods for calculating scattering amplitudes, primarily in gauge theories but also in more general quantum field theory settings. The progress was initiated by Witten's proposal [1] (based partly on an earlier insight by Nair [2]) that $\mathcal{N} = 4$ Super Yang-Mills was dual to a topological string theory in twistor space. This led Cachazo, Svrcek and Witten [3] to the conjecture that (as will be elaborated on in section 3) tree-level amplitudes of gluons may be calulated by sewing together certain on-shell amplitudes using scalar propagators. This procedure has since been known as the CSW rules.

The CSW conjecture has led to several developments in tree-level calculations [4 − 18], and it was also used for some one-loop calculations in an approach developed by Brandhuber, Spence and Travaglini [19 − 22]. It was somewhat unexpected that the rules did apply to loop calculations, but the discrepancy was resolved by the introduction of the 'holomorphic anomaly' [23]. When combined with the unitarity method of Bern, Dixon, Dunbar and Kosower [24 − 26] the concept led to a method for calculating one-loop amplitudes in gauge theory as well as to extensive studies of the twistor space structure of one-loop amplitudes [27 − 31, 26].

Inspired by this, but without depending on it, Britto, Cachazo and Feng showed that, through the concept of generalized unitarity, the calculation of one-loop amplitudes in $\mathcal{N} = 4$ Super Yang-Mills could be reduced to the calculation of certain on-shell tree amplitudes [32]. Some applications are [33 − 36]. When using infrared concistency conditions on the one-loop amplitudes it is possible to obtain compact expressions for tree amplitudes [37, 26, 38], and this was used by the authors mentioned above to develop recursion relations [39] for tree amplitudes. A proof of the latter was developed together with Witten [40], giving rise to the term BCFW recursion. This is elaborated on in section 4.

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The methods of BCFW recursion have been applied and extended with great success to both tree-level calculations in Yang-Mills and other theories [41 – 48] as well as one-loop calculations [49 – 52], and, in some of these cases, been merged with technology such as unitarity methods (see above).

The development described here has taken place alongside a development in string theory and these two have interacted and inspired each other. The string theory development and its interaction with the one described here, is reviewed in [53].

The purpose of the present letter is to prove the CSW rules for gluons in external states using a method closely related to BCFW recursion. This will tie together the two main developments in tree-level calculations described here and show that one follows from the other. In addition, it may shed some light on possible generalizations of the CSW rules.

The contents of this paper are as follows: In section 2 we review our notation and conventions. In sections 3 and 4 we review the CSW and BCFW methods before proceding to the proof of the CSW rules in section 5. In section 6 the implications of the proof are discussed.

2. Notation and conventions

We will be using the notion of colour ordering and the spinor helicity notation. These matters are very well described in [54] whose conventions we use. For the spinor helicity notation, we use the shorthand notation

$$\langle j^+|\not{K}|i^+\rangle = \langle i^-|\not{K}|j^-\rangle = \langle iKj] \,, \tag{2.1}$$

$$\langle i^-| = \langle i| \,, \qquad \langle i^+| = [i| \,, \qquad |i^+\rangle = |i\rangle \,, \qquad |i^-\rangle = |i] \,. \tag{2.2}$$

The derivations all rely on the underlying space being complexified Minkowski space. This means that we may choose the two spinors $|i\rangle$ and $|i]$ associated with k_i^μ to be complex and independent. By convention, $|i\rangle$ is referred to as the holomorphic spinor and $|i]$ is known as the anti-holomorphic spinor.

3. The CSW rules

The starting point of the CSW rules is the amplitudes [55, 56],

$$A(1^\pm, 2^+, 3^+, \ldots, n^+) = 0 \,, \tag{3.1}$$

$$A(1^+, 2^+, \ldots, r^-, \ldots, s^-, \ldots, n^+) = i\frac{\langle rs\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle} \,. \tag{3.2}$$

The latter is a 'maximally helicity violating' (MHV) amplitude whose main characteristic, in this context, is that it depends only on the holomorphic spinors of the incoming particles.

The CSW rules state how an amplitude can be calculated as the sum of contributions each represented by a diagram similar to a Feynman diagram. The vertices of such a diagram have two negative helicity gluons and any number of positive helicity gluons, and the edges connect gluons of negative and positive helicity. The contribution of such a diagram is given as the product of the on-shell amplitudes (3.2) for each MHV vertex and the product of usual scalar propagators for each edge. The entire amplitude is the sum of all possible CSW diagrams. Any CSW diagram contibuting to an amplitude with n negative helicity gluons (an N^{n-2}MHV amplitude, N for 'next-to-') contains $n-2$ propagators, and every diagram is exactly characterized by its propagators.

The internal lines are not lightlike, and thus have no *a priori* spinors to put into (3.2) at each vertex. Instead, one choses a lightlike momentum η^μ and writes the internal momentum, say P_i, as

$$P_i^\mu = P_i^{\flat\mu} + z\eta^\mu, \qquad z = \frac{P_i^2}{2P_i \cdot \eta}, \tag{3.3}$$

such that $P_i^{\flat\mu}$ is lightlike. It is the spinors of this momentum which are used for the vertices. Because the expressions always turn out invariant under scaling of the internal spinors, we may use $|P_i\eta]$ as the holomorphic spinor and $|P_i\eta\rangle$ as the anti-holomorphic spinor. Notice, however, that the latter is never used in the CSW rules, and the end expression is independent of $|\eta\rangle$. Notice also, that even though we choose a displaced version of the internal momenta for defining the vertices, the momenta appearing in the propagators are undisplaced.

4. BCFW recursion

BCFW on-shell recursion is a way to reconstruct a scattering amplitude from its singularities. In its pure form, it consists of adding a momentum $z\eta^\mu$ on one external particle and subtracting it on another while choosing η^μ such that this shift of the two momenta leaves them on-shell. z is a complex variable. This defines a shifted amplitude $\widehat{A}(z)$ where we would like to know $\widehat{A}(0)$. Provided that $\widehat{A}(z) \to 0$ as $z \to \infty$ we can write

$$0 = \frac{1}{2\pi i}\oint_{\mathcal{C}\ \text{at}\ \infty} \frac{\widehat{A}(z)}{z} = \widehat{A}(0) - \sum_{\text{poles}\ z_i} \frac{\mathrm{Res}_{z_i}\widehat{A}(z)}{z_i}\,, \tag{4.1}$$

By inspecting the Feynman rules for a shifted amplitude, we see that there are two sources of poles in the shifted amplitude. The one is the external polarizations where the spinor of an external particle appears in the denominator. This type of pole must be removable since choosing η itself as the reference momentum for the polarization removes the z dependence of the propagator. We may still make an other choice of polarization reference momentum and use that to lower the power of asymptotic z dependence of the shifted amplitude, but we need not consider the resultant pole in (4.1). The other source is the propagators. There may be cases where the singularity coming from a propagator is not physical (i.e., removable) but in that case the residue is zero and it makes no difference whether we include the contribution or not. In conclusion, the poles in z we need to consider are those that arise from propagators which are affected by the shift. It is easily seen that the residue at such a pole is proportional to the product of the on-shell amplitudes at each end of the propagator in question, or

$$A(0) = \sum_{i\in\mathcal{P}} \widehat{A}^{L,i}\left(-\frac{P_i^2}{2P_i\cdot\eta}\right)\frac{i}{P_i^2}\widehat{A}^{R,i}\left(-\frac{P_i^2}{2P_i\cdot\eta}\right), \tag{4.2}$$

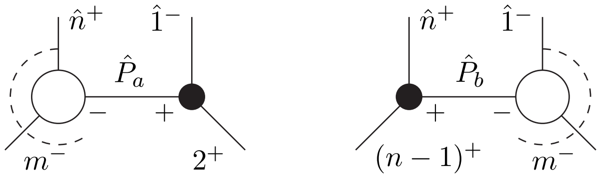
where $\mathcal{P}$ is the set of shifted propagators and $\widehat{A}^{L,i}(z)$, $\widehat{A}^{R,i}(z)$ are the two shifted amplitudes separated by P_i. In this expression, the particles entering and leaving the propagator must be of opposite helicity. We choose to regard two propagators with opposite helicity assignment as distinct although they have the same momentum running through them. Since they never appear in the same diagram, this will not cause problems for the arguments that follow.

The above discussion can be made concrete by a rederivation of (3.2). Consider the amplitude $A(1^-, 2^+, \ldots, m^-, \ldots, n^+)$ and choose the reference momentum to be $|1\rangle[n|$ such that

$$|\hat{1}\rangle[\hat{1}| = |1\rangle([1| + z[n|)\,, \tag{4.3}$$

$$|\hat{n}\rangle[\hat{n}| = (|n\rangle - z|1\rangle)[n|\,. \tag{4.4}$$

In other words, $|1\rangle$ and $|n]$ are unchanged. As $z \to \infty$, the shifted amplitude goes as z^{-1} because the worst Feynman diagram, containing only three-vertices, contributes z from vertices and propagators and z^{-1} from each of the polarizations. Most terms in the sum (4.2) are in fact zero since one of the shifted amplitudes always falls in the category of (3.1). That equation, however, does not hold exactly for complex momenta, as the amplitude $A(1^+, 2^+, 3^-)$ is only zero if one of the square brackets between two of the three particles is zero. The two diagrams which contain this so-called googly 3-vertex are



In the first of these, $\hat{P}_a^2 = (\hat{p}_1 + p_2)^2 = [\hat{1}2]\langle 2\hat{1}\rangle = 0$ requires either $[\hat{1}2]$ or $\langle 21\rangle$ to be 0. Since $\langle 21\rangle$ is externally defined, it must be $[\hat{1}2]$, and hence, the googly amplitude vanishes. In the second diagram an identical argument means that $\langle \hat{n}(n-1)\rangle$ must vanish. That does not interfere with the 3-point googly vertex, but the 3-point MHV would be zero in that case (as it is for real momenta). Notice, that the shifted 3-point googly disappears when the shift is made in the anti-holomorphic spinor and the shifted 3-point MHV disappears when the shift is made in the holomorphic spinor.

Assuming that (3.2) applies to an amplitude with $n-1$ external gluons, we get for the amplitude in question

$$\begin{aligned}
A(0) &= \frac{i\langle 1m\rangle^4}{\langle \hat{P}_b 1\rangle\langle 12\rangle\cdots\langle (n-2)\hat{P}_b\rangle}\frac{i}{P_b^2}\frac{i[n(n-1)]^4}{[\hat{P}_b(n-1)][(n-1)n][n\hat{P}_b]} \\
&= i\frac{\langle 1m\rangle^4}{\langle 12\rangle\cdots\langle (n-3)(n-2)\rangle}\frac{[n(n-1)]^3}{\langle 1\hat{P}_b(n-1)]\langle (n-2)\hat{P}_b n]}\frac{1}{[n(n-1)]\langle (n-1)n\rangle} \\
&= i\frac{\langle 1m\rangle^4}{\langle 12\rangle\cdots\langle (n-3)(n-2)\rangle\langle (n-1)n\rangle}\frac{[(n-1)n]^2}{\langle 1n\rangle[n(n-1)]\langle (n-2)(n-1)\rangle[(n-1)n]} \\
&= i\frac{\langle 1m\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}\,.
\end{aligned} \tag{4.5}$$

This provides a proof by induction of (3.2). In this derivation we never needed to use the actual value of z_i, something which we will need in general.

Before proceding to the direct proof of the CSW rules, it should be noted that BCFW recursion can reconstruct any gluon amplitude solely from the knowledge of its singularities. Since the CSW rules are known to provide results which are Lorentz and gauge invariant and which have the right singularities, the existence of BCFW recursion provides an indirect proof of the CSW rules, as was noted in [40].

5. Proof

We prove the CSW rules by induction. First, we choose a shift of the external momenta which will allow us to prove that an NMHV amplitude may be calculated by means of the CSW rules. We then employ a generalized version of that shift to a N^{n-1}MHV amplitude in order to show that we may calculate it by the CSW rules provided that we can calculate all more helicity violating amplitudes by the CSW rules.

Instead of choosing shifts which minimize the terms to sum over in (4.2), we choose shifts that affect every propagator which may occur in a CSW diagram. A propagator is characterized by a consecutive set of external particles $(i, \ldots, j)$ whose total momentum runs through it, and the propagator occurs in a CSW diagram if (and only if) the set and its compliment include at least one gluon of negative helicity each. Exactly this set of propagators is affected by a shift on every negative helicity gluon, as long as the sum of any subset of the shifts does not vanish. In addition, if all 3-point googly amplitudes are to drop out, the shifts must all involve the anti-holomorphic spinors as was seen in the previous section.

For the NMHV amplitude, these arguments suggest the choice,

$$\begin{aligned} |\hat{m}_1\rangle[\hat{m}_1| &= |m_1\rangle([m_1| + z\langle m_2 m_3\rangle[\eta|)\,, \\ |\hat{m}_2\rangle[\hat{m}_2| &= |m_2\rangle([m_2| + z\langle m_3 m_1\rangle[\eta|)\,, \\ |\hat{m}_3\rangle[\hat{m}_3| &= |m_3\rangle([m_3| + z\langle m_1 m_2\rangle[\eta|)\,, \end{aligned} \tag{5.1}$$

where $m_{1,2,3}$ are the three negative helicity gluons. The sum of the momentum shifts vanishes by the Schouten identity. As $z \to \infty$, the most dangerous Feynman diagram, composed of three-vertices only, goes as z^{-2} because the vertices and propagators contribute a z while the polarizations contribute z^{-3}. Thus, all conditions for performing BCFW recursion with this shift are satisfied.

The result of this is evidently the sum of all propagators which may appear in the CSW diagram times two MHV amplitudes, one for each end of the propagator, both of which are shifted in such a way that the propagator goes on-shell,

$$\widehat{A}_3(0) = \sum_{i \in \mathcal{P}_3} \widehat{A}_2^{1,i}\left(-\frac{P_i^2}{2\langle\lambda_i P_i \eta]}\right) \frac{i}{P_i^2} \widehat{A}_2^{2,i}\left(-\frac{P_i^2}{2\langle\lambda_i P_i \eta]}\right). \tag{5.2}$$

In this expression $\langle\lambda_i|$ is short for the spinor multiplying $z[\eta|$ in the shifted propagator momentum; in the NMHV case, all $\langle\lambda_i|$ are equal to $\langle m_1 m_2\rangle\langle m_3|$ or one of its cyclic permutations. The subscripts on the A's denote the number of negative helicity gluons.

The shifted MHV amplitudes of the expression are completely holomorphic, and since the shift is purely anti-holomorphic, they are not influenced. The momentum running in the propagator i has been put on-shell using the reference momentum $|\lambda_i\rangle[\eta|$, and it takes part in the MHV vertices in the same manner as in the CSW rules. Hence its holomorohic spinor can be written as $|P_i\eta]$ as in those rules. By collecting all these facts, it is seen that an NMHV amplitude may indeed be calculated by the CSW rules.

Let us now assume that the CSW rules hold for any N^pMHV amplitude when $p < n-1$ and consider an N^{n-1}MHV amplitude. We now shift the momenta of all the $n+1$ negative helicity gluons as

$$|\hat{m}_i\rangle[\hat{m}_i| = |m_i\rangle([m_i| + zr_i[\eta|)\,, \tag{5.3}$$

where $\sum_{i=1}^{n+1} |m_i\rangle r_i = 0$ without the same holding for any subset of them. By the same argument as for NMHV, the shifted amplitude goes as z^{-n} as $z \to \infty$. The z-dependent propagators are exactly those that appear in the CSW rules, and the z integration is frozen on each of these in turn,

$$A_{n+1}(0) = \sum_{i\in\mathcal{P}_{n+1}} \widehat{A}^{L,i}_{m(i)}\left(-\frac{P_i^2}{\langle\lambda_i P_i\eta]}\right)\frac{i}{P_i^2}\widehat{A}^{R,i}_{n+2-m(i)}\left(-\frac{P_i^2}{\langle\lambda_i P_i\eta]}\right). \tag{5.4}$$

Again $|\lambda_i\rangle$ denotes the sum of $|m_j\rangle r_j$ over the momentum shifts isolated on one side of the propagator. In this expression, the $\widehat{A}$'s have a lower number of negative helicity gluons than the original amplitude.

Because of our induction assumption we may calculate the $\widehat{A}$'s using the CSW rules, in which case (5.4) becomes a sum of terms each consisting of n MHV vertices, one unshifted propagator, and $n-2$ propagators shifted as to put the unshifted one on-shell. In the same way we argued above, we may take $|P_i\eta]$ to be the holomorphic spinor of $\widehat{P}_i$, and as above, the MHV vertices are independent of the shift. This leads us to conclude that every term in the above-mentioned expansion of (5.4) represents a CSW diagram for the N^{n-1}MHV amplitude where the MHV vertices are equal to those given by the CSW rules while all but one of the propagators are shifted.

Every N^{n-1}MHV CSW diagram appears many times in this way but with different propagators unshifted. Let us consider one N^{n-1}MHV CSW diagram and extract all terms in (5.4) where it appears. Since the sum runs over all possible CSW propagators, we obtain contributions exactly from the part of the sum which runs over the $n-1$ propagators of the CSW diagram in question. For each of these, there are unique CSW decompositions of the two shifted amplitudes which yield the set of propagators we seek. The contributions to the diagram from (5.4) thus sums up to the MHV vertices prescribed by the CSW rules times

$$\sum_{i=1}^{n-1}\frac{i}{P_i^2}\prod_{j=1,j\neq i}^{n-1}\frac{i}{\widehat{P}_{ji}^2}\,, \tag{5.5}$$

where

$$\widehat{P}_{ji}^2 = (P_j + z_i|\lambda_j\rangle[\eta|)^2 = P_j^2 - P_i^2\frac{\langle\lambda_j P_j\eta]}{\langle\lambda_i P_i\eta]} \tag{5.6}$$

is propagator j when its momentum is shifted such that propagator i goes on-shell. Equation (5.5) however, is just the product of unshifted propagators. This can be realized by making the shift

$$\prod_{i=1}^{n-1} \frac{i}{P_i^2} \to \prod_{i=1}^{n-1} \frac{i}{(P_i + z|\lambda_i\rangle[\eta|)^2}\,, \tag{5.7}$$

and using (4.1) in the same way as to derive (4.2). Thus, the contribution from the CSW diagram considered is exactly as given by the CSW rules.

We can apply this argument to any N^{n-1}MHV CSW diagram. Since we have also shown that every part of (5.4) can be written in a way where it contributes to an N^{n-1}MHV CSW diagram as described, it follows that the N^{n-1}MHV CSW rules are in one-to-one correspondence with (5.4). By induction, any amplitude can thus be calculated by the CSW rules.

6. Discussion

As noted at the end of section 4, the CSW rules have been known to give the right results, but only by an indirect proof. The present proof provides an explanation of why the rules hold true, and what is their connection to BCFW recursion.

The proof also clarifies a point in the formulation of the CSW rules. When constructing the internal spinors through (3.3), η^μ is assumed to be constant throughout the calculation, even though this is only strictly necessary for the anti-holomorphic spinor. The holomorphic spinor $|\eta\rangle$ may differ significantly from propagator to propagator.

Although this fact does not alter tree-level Yang-Mills calculations, it may have implications in extensions of the CSW rules. When trying to develop a CSW approach to a field theory where (the equivalent of) MHV amplitudes are not holomorphic (e.g., gravity) there will be a dependence on the holomorphic reference spinor which might not be assumed to be constant. Also, when the CSW rules are applied to loop calculations, the full reference momentum takes part in the expressions, and a precise knowledgement of the possible holomorphic reference spinors is needed. The fact that the MHV one-loop calculations *do* carry through in a formulation where the holomorphic reference spinor is unconstrained [19–22], shows that there is probably more to the CSW rules than suggested by this proof.

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References

[1] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[2] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215.

[3] F. Cachazo, P. Svrcek and E. Witten, *Mhv vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [hep-th/0403047].

[4] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *JHEP* **05** (2004) 070 [hep-th/0404072].

[5] G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [hep-th/0407027].

[6] J.-B. Wu and C.-J. Zhu, *Mhv vertices and scattering amplitudes in gauge theory*, *JHEP* **07** (2004) 032 [hep-th/0406085].

[7] J.-B. Wu and C.-J. Zhu, *Mhv vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos*, *JHEP* **09** (2004) 063 [hep-th/0406146].

[8] X. Su and J.-B. Wu, *Six-quark amplitudes from fermionic MHV vertices*, *Mod. Phys. Lett.* **A 20** (2005) 1065 [hep-th/0409228].

[9] I. Bena, Z. Bern and D.A. Kosower, *Twistor-space recursive formulation of gauge theory amplitudes*, *Phys. Rev.* **D 71** (2005) 045008 [hep-th/0406133].

[10] D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D 71** (2005) 045007 [hep-th/0406175].

[11] S. Giombi, R. Ricci, D. Robles-Llana and D. Trancanelli, *A note on twistor gravity amplitudes*, *JHEP* **07** (2004) 059 [hep-th/0405086].

[12] L.J. Dixon, E.W.N. Glover and V.V. Khoze, *Mhv rules for Higgs plus multi-gluon amplitudes*, *JHEP* **12** (2004) 015 [hep-th/0411092].

[13] Z. Bern, D. Forde, D.A. Kosower and P. Mastrolia, *Twistor-inspired construction of electroweak vector boson currents*, *Phys. Rev.* **D 72** (2005) 025006 [hep-ph/0412167].

[14] S.D. Badger, E.W.N. Glover and V.V. Khoze, *MHV rules for Higgs plus multi-parton amplitudes*, *JHEP* **03** (2005) 023 [hep-th/0412275].

[15] T.G. Birthwright, E.W.N. Glover, V.V. Khoze and P. Marquard, *Multi-gluon collinear limits from MHV diagrams*, *JHEP* **05** (2005) 013 [hep-ph/0503063].

[16] T.G. Birthwright, E.W.N. Glover, V.V. Khoze and P. Marquard, *Collinear limits in QCD from MHV rules*, *JHEP* **07** (2005) 068 [hep-ph/0505219].

[17] P. Marquard and T.G. Birthwright, *Multi gluon collinear limits from MHV amplitudes*, hep-ph/0505264.

[18] C. Schwinn and S. Weinzierl, *Scalar diagrammatic rules for born amplitudes in QCD*, *JHEP* **05** (2005) 006 [hep-th/0503015].

[19] A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from MHV vertices*, *Nucl. Phys.* **B 706** (2005) 150 [hep-th/0407214].

[20] C. Quigley and M. Rozali, *One-loop MHV amplitudes in supersymmetric gauge theories*, *JHEP* **01** (2005) 053 [hep-th/0410278].

[21] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A twistor approach to one-loop amplitudes in $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 706** (2005) 100 [hep-th/0410280].

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[22] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [hep-th/0412108].

[23] F. Cachazo, P. Svrcek and E. Witten, *Gauge theory amplitudes in twistor space and holomorphic anomaly*, *JHEP* **10** (2004) 077 [hep-th/0409245].

[24] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].

[25] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265].

[26] Z. Bern, L.J. Dixon and D.A. Kosower, *All next-to-maximally helicity-violating one-loop gluon amplitudes in $N = 4$ super- Yang-Mills theory*, *Phys. Rev.* **D 72** (2005) 045014 [hep-th/0412210].

[27] F. Cachazo, *Holomorphic anomaly of unitarity cuts and one-loop gauge theory amplitudes*, hep-th/0410077.

[28] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, *Phys. Rev.* **D 71** (2005) 025012 [hep-th/0410179].

[29] S.J. Bidder, N.E.J. Bjerrum-Bohr, L.J. Dixon and D.C. Dunbar, *$N = 1$ supersymmetric one-loop amplitudes and the holomorphic anomaly of unitarity cuts*, *Phys. Lett.* **B 606** (2005) 189 [hep-th/0410296].

[30] R. Britto, F. Cachazo and B. Feng, *Coplanarity in twistor space of $N = 4$ next-to-MHV one-loop amplitude coefficients*, *Phys. Lett.* **B 611** (2005) 167 [hep-th/0411107].

[31] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *Twistor space structure of the box coefficients of $N = 1$ one-loop amplitudes*, *Phys. Lett.* **B 608** (2005) 151 [hep-th/0412023].

[32] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super- Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103].

[33] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *One-loop gluon scattering amplitudes in theories with $n < 4$ supersymmetries*, *Phys. Lett.* **B 612** (2005) 75 [hep-th/0502028].

[34] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, *JHEP* **10** (2005) 011 [hep-th/0506068].

[35] E.I. Buchbinder and F. Cachazo, *Two-loop amplitudes of gluons and octa-cuts in $N = 4$ super Yang-Mills*, hep-th/0506126.

[36] K. Risager, S.J. Bidder and W.B. Perkins, *One-loop nMHV amplitudes involving gluinos and scalars in $N = 4$ gauge theory*, *JHEP* **10** (2005) 003 [hep-th/0507170].

[37] Z. Bern, V. Del Duca, L.J. Dixon and D.A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in $N = 4$ super- Yang-Mills theory*, *Phys. Rev.* **D 71** (2005) 045006 [hep-th/0410224].

[38] R. Roiban, M. Spradlin and A. Volovich, *Dissolving $N = 4$ loop amplitudes into QCD tree amplitudes*, *Phys. Rev. Lett.* **94** (2005) 102002 [hep-th/0412265].

[39] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308].

[40] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052].

[41] M.-x. Luo and C.-k. Wen, *Recursion relations for tree amplitudes in super gauge theories*, *JHEP* **03** (2005) 004 [hep-th/0501121].

[42] M.-x. Luo and C.-k. Wen, *Compact formulas for all tree amplitudes of six partons*, *Phys. Rev.* **D 71** (2005) 091501 [hep-th/0502009].

[43] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A recursion relation for gravity amplitudes*, *Nucl. Phys.* **B 721** (2005) 98 [hep-th/0502146].

[44] F. Cachazo and P. Svrcek, *Tree level recursion relations in general relativity*, hep-th/0502160.

[45] R. Britto, B. Feng, R. Roiban, M. Spradlin and A. Volovich, *All split helicity tree-level gluon amplitudes*, *Phys. Rev.* **D 71** (2005) 105017 [hep-th/0503198].

[46] S.D. Badger, E.W.N. Glover, V.V. Khoze and P. Svrcek, *Recursion relations for gauge theory amplitudes with massive particles*, *JHEP* **07** (2005) 025 [hep-th/0504159].

[47] S.D. Badger, E.W.N. Glover and V.V. Khoze, *Recursion relations for gauge theory amplitudes with massive vector bosons and fermions*, hep-th/0507161.

[48] D. Forde and D.A. Kosower, *All-multiplicity amplitudes with massive scalars*, hep-th/0507292.

[49] Z. Bern, L.J. Dixon and D.A. Kosower, *On-shell recurrence relations for one-loop QCD amplitudes*, *Phys. Rev.* **D 71** (2005) 105013 [hep-th/0501240].

[50] Z. Bern, L.J. Dixon and D.A. Kosower, *The last of the finite loop amplitudes in QCD*, hep-ph/0505055.

[51] Z. Bern, L.J. Dixon and D.A. Kosower, *Bootstrapping multi-parton loop amplitudes in QCD*, hep-ph/0507005.

[52] Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, hep-ph/0507019.

[53] F. Cachazo and P. Svrcek, *Lectures on twistor strings and perturbative Yang-Mills theory*, *PoS* **RTN2005** (2005) 004 [hep-th/0504194].

[54] L.J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.

[55] S.J. Parke and T.R. Taylor, *An amplitude for n gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[56] F.A. Berends and W.T. Giele, *Recursive calculations for processes with n gluons*, *Nucl. Phys.* **B 306** (1988) 759.

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